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The scattering of electroelastic waves by an ellipsoidal inclusion in piezoelectric medium

Hao Ma [a,b], Biao Wang [b,*]

[a] *School of Science, QingDao Technological University, QingDao 266033, P.R. China*
[b] *Center for Composite Materials, Harbin Institute of Technology, Harbin 150001, P.R. China*

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Abstract

The scattering problem for a single ellipsoidal piezoelectric inclusion embedded in piezoelectric medium is investigated. Based on the polarization method, the extended displacements are expressed in terms of integral equations, whose kernels are obtained from the Green's functions of homogenous matrix. In this paper, the 3D dynamic Green's functions are derived by means of the Radon transform technique. To illustrate the use of the equations, scattering by a piezoelectric, ellipsoidal inhomogeneity in a piezoelectric medium is considered in the low frequency and an asymptotic formula for this scattering cross-section is obtained. Numerical results of the scattering cross-sections are carried out for a spheroidal $BaTiO_3$-inclusion in a PZT-5H-matrix.

1. Introduction

For recent two decades, various types of piezoelectric composites have been developed and widely applied in many engineering applications, for example, sonar projector, underwater acoustic and medical ultrasonic imaging, etc. In general, comprising two or more constituents, piezoelectric composites take advantage of each constituent and have superior electromechanical coupling characteristics compared to homogenous piezoelectric material. These materials have been fabricated in many forms including the second phase piezoelectric inclusions embedded in a polymer matrix and polymer inclusions in a solid piezoelectric ceramic matrix. The second-phase piezoelectric inclusions in the composites can be continuous fibers, short fibers, holes, voids or dispersed quasispherical particles. The studies on the physical and

* Corresponding author. Tel./fax: +86 5325071560.
E-mail addresses: ma670315@263.net (H. Ma), wangbiao@hit.edu.cn (B. Wang).

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mechanical properties of such materials become very important in engineering. By use of different methods, many researchers have adequately investigated the problems for the piezoelectric composite containing inclusions.

Deeg (1980) extended the classical method of Eshelby (1957) to the problem of an ellipsoidal inclusion embedded in piezoelectric matrix. In case of ellipsoidal inclusion and in the limiting case of elliptical crack, Wang (1992a,b) represented the coupled electroelastic fields in integral form, whose kernels were the Green's functions, and obtained the analytical solutions for the coupled elastic and electric field inside the inclusion and just outside the inclusion. As a limiting case of an ellipsoidal inclusion, Dunn and Taya (1993) obtained close-form Eshelby tensor for elliptic cylindrical, circular cylindrical and ribbon-like inclusion. Kuo and Huang (1997) considered the problems of piezoelectric composites containing spatially oriented inclusions. The explicit expressions for the electrostatic tensors analogous to the Eshelby tensors were obtained, and with these tensors, the analytical expressions for the electroelastic fields had been derived.

All the results mentioned above are static solutions. Unlike static problems, relatively little work has been done regarding on the wave propagation in the inhomogeneous piezoelectric solid. Maurizio Romeo (2002) considered the propagation of transient shear horizontal waves in the piezoelectric layer with free boundaries within a time domain approach, and using the separation of space variables, the problem for the Laplace transforms of electromechanical fields was solved. Levin et al. (2002) investigated the propagation of electroacoustic waves in a piezoelectric transversely isotropic medium containing a single inhomogeneity fiber. By means of Green's function approach, a system of coupled integral equations for the electroelastic field was solved in closed form in the long-wave approximation. The objective of the present paper is to provide the general solutions for the electroelastic field for the scattering problem by a single ellipsoidal piezoelectric inclusion embedded in a piezoelectric medium. The present method is based on the polarization approach, which has been adopted by Willis (1980) to deal with the scattering problem in anistropic elastic body.

The paper is arranged as follows: in Section 2, the integral equations for the electroelastic dynamic problem are derived for the inhomogeneous piezoelectric body. The kernels of the integral equations are obtained from the Green's function of the 'comparison body', which is identified with the matrix in scattering problem. In Section 3, by the use of the Radon transform method, three-dimensional dynamic Green's functions for the homogeneous piezoelectric solids are obtained. The Green's functions can be represented as a summation of a regular dynamic term and a singular static term. In Section 4, the expression for the scattering cross-section of the inclusion in the piezoelectric mediums is given. In Section 5, with the retention of just the terms of lowest order in the series for the two polarizations, a formal solution in the low frequency is developed for the scattering problem of an ellipsoidal inclusion in the piezoelectric matrix. Particular attention is paid to the scattering cross-section and its approximate solution is given. In Section 6, numerical results are given for a single spheroidal $BaTiO_3$-inclusion in a PZT-5H-matrix, including the limiting cases for the scattering cross-sections of a flat disc, rigid circular disc, long fiber and rigid fiber.

2. Polarization equations

For inhomogeneous piezoelectric media, the constitutive equations can be written as

$$\begin{aligned} \sigma_{ij}(\vec{x},t) &= c_{ijkl}(\vec{x},t)u_{k,l}(\vec{x},t) + e_{lij}(\vec{x},t)\varphi_{,l}(\vec{x},t) \\ D_i(\vec{x},t) &= e_{ikl}(\vec{x},t)u_{k,l}(\vec{x},t) - \varepsilon_{il}(\vec{x},t)\varphi_{,l}(\vec{x},t) \end{aligned} \tag{2.1}$$

where $\vec{x} = (x_1, x_2, x_3)$, $\mathbf{c}$ is the elastic moduli tensor, $\mathbf{e}$ is the piezoelectric moduli tensor and $\boldsymbol{\varepsilon}$ is the permittivity of the dielectric material. $\mathbf{u}$ and $\boldsymbol{\varphi}$ are the elastic displacement and the electric potential. $\mathbf{D}$ and $\boldsymbol{\sigma}$ are the electric displacement and the elastic stress tensor, respectively.

It is noteworthy that the material constants satisfy the following symmetric relations

$$\begin{aligned} & c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} \\ & e_{lij} = e_{lji}, \quad \varepsilon_{il} = \varepsilon_{li} \end{aligned} \tag{2.2}$$

The substitution of Eq. (2.1) into the elastic field equations and Maxwell's equation leads to a coupled system of differential equations as follows:

$$\begin{aligned} & (c_{ijkl}u_{k,l} + e_{lij}\varphi_{,l})_i + f_j = (\rho_{jk}u_{k,t})_{,t} \\ & (e_{ikl}u_{k,l} - \varepsilon_{il}\varphi_{,l})_{,i} - q = 0 \end{aligned} \tag{2.3}$$

in which $\boldsymbol{\rho}$ is the material's density, $\mathbf{f}$ is the body force tensor, q is density of free electric charges.

In order to simply the formulations presented above, the following notations are introduced (Pan and Tonon, 2000),

$$u_I = \begin{cases} u_i & I = 1,2,3 \\ \varphi & I = 4 \end{cases} \tag{2.4}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij} & J = 1,2,3 \\ D_i & J = 4 \end{cases} \tag{2.5}$$

$$C_{iJKl} = \begin{cases} c_{ijkl} & J,K = 1,2,3 \\ e_{lij} & J = 1,2,3; K = 4 \\ e_{ikl} & J = 4; K = 1,2,3 \\ -\varepsilon_{il} & J,K = 4 \end{cases} \tag{2.6}$$

$$F_J = \begin{cases} f_j & J = 1,2,3 \\ -q & J = 4 \end{cases} \tag{2.7}$$

$$\rho_{JK} = \begin{cases} \rho_{jk} & J,K = 1,2,3 \\ 0 & J = 4 \text{ or } K = 4 \end{cases} \tag{2.8}$$

where u_I, σ_{iJ}, C_{iJKl}, F_J and ρ_{JK} are called the extended displacements, extended stress, extended body force and extended momentum, respectively. It is note that in above Eqs. (2.4)–(2.8), and late on, the lowercase and uppercase subscripts take on the range 1–3 and 1–4, respectively. In terms of this shorthand notation, the equations of wave motion can be rewritten as

$$(C_{iJKl}u_{K,l})_{,i} + F_J = (\rho_{JK}u_{K,t})_{,t} \tag{2.9}$$

together with suitable initial and boundary conditions. The initial value problem will be considered for a body occupying a region V, while the initial conditions will be that extended displacement $\mathbf{u}$ and momentum $\boldsymbol{\rho}$ are prescribed throughout V at $t = 0$, and the boundary conditions will be of standard type such as: either the extended force or extended displacement will be prescribed over ∂V for all positive t.

In inhomogeneous piezoelectric medium, the solution of equation (2.9) is very difficult to be obtained directly. Now consider, a "comparison" body, occupying the same region V but having operator $\mathbf{C^0}$ and $\boldsymbol{\rho}^\mathbf{0}$. Substitution into the equation of motion (2.9) yields

$$(C^0_{iJKl}u_{K,l})_{,i} + F_J = (\rho^0_{JK}u_{K,t})_{,t} \tag{2.10}$$

It is useful to consider an adjoint problem for the field $\mathbf{v}$ and the corresponding "adjoint" operators are $\mathbf{C}^*$ and $\boldsymbol{\rho}^*$, then gives

$$(C^*_{lKJi}v_{J,i})_{,l} + E_K = (\rho^*_{KJ}v_{J,i})_{,t} \tag{2.11}$$

where E_K is the extended body force of an adjoint problem for the field **v**.

By use of Gauss' theorem, Eqs. (2.10) and (2.11) leads to the identity

$$\begin{aligned}&\int_0^\infty \mathrm{d}t \int_{\partial V} \mathrm{d}S[v_J C^0_{iJKl}u_{K,l}n_i - u_K C^*_{lKJi}v_{J,i}n_l] + \int_0^\infty \mathrm{d}t \int_V \mathrm{d}\vec{x}(v_J F_J - u_K E_K)\\ &\quad = \int_V \mathrm{d}\vec{x}[v_J\rho^0_{JK}u_{K,t} - u_K\rho^*_{KJ}v_{J,t}]^\infty_{t=0}\end{aligned} \tag{2.12}$$

In the derivation of Eq. (2.12), the relations

$$\begin{aligned}&\int_0^\infty \mathrm{d}t \int_V \mathrm{d}\vec{x}(v_{J,i}C^0_{iJKl}u_{K,l} - u_{K,l}C^*_{lKJl}v_{J,i}) = 0\\ &\int_0^\infty \mathrm{d}t \int_V \mathrm{d}\vec{x}(v_{J,t}\rho^0_{JK}u_{K,t} - u_{K,t}\rho^*_{KJ}v_{J,t}) = 0\end{aligned} \tag{2.13}$$

are used.

Now let **G** be the Green's functions for the comparison body, their components satisfy the equations

$$(C^0_{iJKl}G_{KP,l})_{,i} + \delta_{JP}\delta(\vec{x} - \vec{x}')\delta(t - t') = (\rho^0_{JK}G_{KP,t})_{,t} \tag{2.14}$$

with homogeneous initial and boundary conditions. The first index of $G_{KP}(\vec{x})$ denotes the component of the extended Green's displacements, while the second denotes the direction of the extended point force. The Green's functions represent the coupled elastic and electric response to the application of time-harmonic point force or point charge.

And let **G*** be the adjoint Green's function, having components to satisfy

$$(C^*_{lKJi}G^*_{JQ,i})_{,l} + \delta_{KQ}\delta(\vec{x} - \vec{x}'')\delta(t - t'') = (\rho^*_{KJ}G^*_{JQ,t})_{,t} \tag{2.15}$$

with the corresponding adjoint boundary conditions. Application of identity (2.12) to **G** and **G*** then gives

$$G^*_{PQ}(\vec{x}', t', \vec{x}'', t'') = G_{PQ}(\vec{x}'', t'', \vec{x}', t') \tag{2.16}$$

The above equation shows that the useful adjoint Green's function **G*** may be obtained from the Green's function **G**, which has more direct physical meaning.

With this background, we employ the extended field **u** that exists in the original body to produce two polarizations

$$\tau_{iJ} = (C_{iJKl} - C^0_{iJKl})u_{K,l}, \quad \pi_J = (\rho_{JK} - \rho^0_{JK})u_{K,t} \tag{2.17}$$

relative to the comparison medium. The extended stress and the extended momentum in the original body may now be given in the forms

$$\sigma_{iJ} = \tau_{iJ} + C^0_{iJKl}u_{K,l}, \quad \rho_{JK}u_{K,t} = \rho^0_{JK}u_{K,t} + \pi_J \tag{2.18}$$

and substitution of (2.18) into the equation of motion (2.9) leads to

$$(C^0_{iJKl}u_{K,l})_{,i} + F_J + \tau_{iJ,i} - \pi_{J,t} = (\rho^0_{JK}u_{K,t})_{,t} \tag{2.19}$$

Application of the identity (2.12) to the field **u** defined by (2.19) and **G*** yields

$$u_Q(\vec{x}'', t'') = -\int \mathrm{d}t \int_V \mathrm{d}\vec{x}\left[G^*_{JQ,i}(\vec{x}, t, \vec{x}'', t'')\tau_{iJ}(\vec{x}, t) - G^*_{jQ,t}(\vec{x}, t, \vec{x}'', t'')\pi_j\right] + u^0_Q(\vec{x}'', t'') \tag{2.20}$$

in which

$$u^0_Q(\vec{x}'',t'') = \int \mathrm{d}t \int_V \mathrm{d}\vec{x}\, G^*_{JQ}(\vec{x},t,\vec{x}'',t'')F_J(\vec{x},t) - \int \mathrm{d}t \int_{\partial V} \mathrm{d}S\Big\{u_K C^*_{lKJi}G^*_{JQ,i}n_l - G^*_{JQ}[C^0_{iJKl}u_{K,l} + \tau_{iJ}]n_i\Big\}$$
$$+ \int_V \mathrm{d}\vec{x}\,\Big\{G^*_{jQ}(\vec{x},0,\vec{x}'',t'')[(\rho^0_{jk}u_{k,l})(\vec{x},0) + \pi_j(\vec{x},0)] - u_k(\vec{x},0)(\rho^*_{kj}G^*_{jQ,t})(\vec{x},0,\vec{x}'',t'')\Big\} \tag{2.21}$$

Eq. (2.20) shows that **u** is the exact solution of the given boundary value problem, but for the comparison body rather than the original; it should be noted that this interpretation is valid only if momentum rather than velocity is regarded as prescribed initially. Comparing Eqs. (2.20) and (2.21) with the Willis's results (Willis, 1980) for the inhomogeneous anisotropic case, it can be proved that the present formations can be reduced to the pure elastic case when the piezoelectric moduli and the permittivity tend to zero, i.e, both the lowercase and uppercase subscripts in Eqs. (2.20) and (2.21) only take on the range 1–3.

Symbolically, therefore

$$\mathbf{u} = -\mathbf{S}\boldsymbol{\tau} - \mathbf{M}\boldsymbol{\pi} + \mathbf{u}^0 \tag{2.22}$$

where

$$(\mathbf{S}\boldsymbol{\tau})_Q(\vec{x},t) = \int \mathrm{d}t' \int_V \mathrm{d}\vec{x}' S_{QiJ}(\vec{x},t,\vec{x}',t')\tau_{iJ}(\vec{x}',t') \tag{2.23}$$

$$(\mathbf{M}\boldsymbol{\pi})_Q(\vec{x},t) = \int \mathrm{d}t' \int_V \mathrm{d}\vec{x}' M_{Qj}(\vec{x},t,\vec{x}',t')\pi_j(\vec{x}',t') \tag{2.24}$$

and

$$S_{QiJ}(\vec{x},t,\vec{x}',t') = \frac{\partial G^*_{JQ}}{\partial x'_i}(\vec{x}',t',\vec{x},t) = \frac{\partial G_{QJ}}{\partial x'_i}(\vec{x},t,\vec{x}',t') \tag{2.25}$$

$$M_{Qj}(\vec{x},t,\vec{x}',t') = -\frac{\partial G^*_{jQ}}{\partial t'}(\vec{x}',t',\vec{x},t) = -\frac{\partial G_{Qj}}{\partial t'}(\vec{x},t,\vec{x}',t') \tag{2.26}$$

Substitution of (2.22) into (2.18) gives the equations

$$(\mathbf{C} - \mathbf{C}^0)^{-1}_{iJQl}\tau_{iJ} + (\mathbf{S}_x)_{QliJ}\tau_{iJ} + (\mathbf{M}_x)_{Qlj}\pi_j = u^0_{Q,l} \tag{2.27}$$

$$(\boldsymbol{\rho} - \boldsymbol{\rho}^0)^{-1}_{jq}\pi_j + (\mathbf{S}_t)_{qiJ}\tau_{iJ} + (\mathbf{M}_t)_{qj}\pi_j = u^0_{q,t} \tag{2.28}$$

where $\mathbf{S}_x$, $\mathbf{M}_x$, $\mathbf{S}_t$ and $\mathbf{M}_t$ are operators with kernels

$$(\mathbf{S}_x)_{QliJ} = \frac{\partial^2 G_{QJ}}{\partial x_l \partial x'_i}, \quad (\mathbf{M}_x)_{Qlj} = -\frac{\partial^2 G_{Qj}}{\partial x_l \partial t'} \tag{2.29}$$

$$(\mathbf{S}_t)_{qiJ} = \frac{\partial^2 G_{qJ}}{\partial t \partial x'_i}, \quad (\mathbf{M}_t)_{qj} = -\frac{\partial^2 G_{qj}}{\partial t \partial t'} \tag{2.30}$$

Eqs. (2.27) and (2.28), together with the formula (2.22), will be applied to the problem of scattering plane wave by a single inclusion.

3. Scattering from an inclusion

In this section, the scattering problem will be considered based on equation (2.27) and (2.28). The matrix will have the operator **C** and $\boldsymbol{\rho}$, while the inclusion have the operator **C*** and $\boldsymbol{\rho}$*. The comparison material will be taken as identical to the matrix, so that the polarizations $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ are non-zero only over the volume V^* occupied by the inclusion. The terms on the right-hand side of Eqs. (2.27) and (2.28) will be associated with the incident plane wave

$$u_K^0 = U_K \mathrm{e}^{-\mathrm{i}[k_0(\vec{n}^0 \bullet \vec{x})+\omega t]} \tag{3.1}$$

where $\mathbf{n}^0$ is unit vector and the polarization $\mathbf{U}$ and wavenumber k satisfy

$$[k_0^2 L_{JK}(\vec{n}^0) - \omega^2 \rho_{JK}]U_K = 0 \tag{3.2}$$

where

$$L_{JK}(\vec{n}^0) = C_{iJKl} n_i^0 n_l^0 \tag{3.3}$$

We consider now a harmonic oscillation in the homogeneous piezoelectric matrix with frequency ω. A steady-state solution will be sought, in which dependent $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ on t only through a factor $\mathrm{e}^{-\mathrm{i}\omega t}$. Correspondingly, time-reduced versions of operator **S**, **M** are required. Because the matrix is infinite, they are obtained from the time-reduced Green's function for an infinite body. The Green's function **G** satisfies

$$C_{iJKl} \frac{\partial^2 G_{KP}(\vec{x})}{\partial x_i \partial x_l} + \omega^2 \rho_{JK} G_{KP}(\vec{x}) + \delta_{JP}\delta(\vec{x}) = 0 \tag{3.4}$$

This equation requires that **G** is an analytic function of ω in the upper half of the complex ω-plane. Since the dynamic 3D Green's functions is not available, the derivation of formulation will be presented in detail.

An application of the Radon transform defined by (A.1) to Eq. (3.4) gives

$$\left[L_{JK}(\vec{n}) \frac{\partial^2}{\partial s^2} + \omega^2 \rho_{JK}\right] \widehat{G}_{KP}(s) + \delta_{JP}\delta(s) = 0 \tag{3.5}$$

where

$$\boldsymbol{\rho} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

Eq. (3.5) can be decomposed as follows:

$$\left[L_{jk}(\vec{n}) \frac{\partial^2}{\partial s^2} + \rho\omega^2 \delta_{jk}\right] \widehat{G}_{kp}(s) + L_{j4}(\vec{n}) \frac{\partial^2 \widehat{G}_{4p}(s)}{\partial s^2} = -\delta_{jp}\delta(s) \tag{3.7}$$

$$\left[L_{jk}(\vec{n}) \frac{\partial^2}{\partial s^2} + \rho\omega^2 \delta_{jk}\right] \widehat{G}_{k4}(s) + L_{j4}(\vec{n}) \frac{\partial^2 \widehat{G}_{44}(s)}{\partial s^2} = 0 \tag{3.8}$$

$$L_{4k}(\vec{n}) \frac{\partial^2 \widehat{G}_{kp}}{\partial s^2} + L_{44}(\vec{n}) \frac{\partial^2 \widehat{G}_{4p}}{\partial s^2} = 0 \tag{3.9}$$

$$L_{4k}(\vec{n}) \frac{\partial^2 \widehat{G}_{k4}}{\partial s^2} + L_{44}(\vec{n}) \frac{\partial^2 \widehat{G}_{44}}{\partial s^2} = -\delta(s) \tag{3.10}$$

After some mathematical operations, the above equations become

$$\left[\Gamma_{jk}(\vec{n}) \frac{\partial^2}{\partial s^2} + \rho\omega^2 \delta_{jk}\right] \widehat{G}_{kp} = -\delta_{jp}\delta(s) \tag{3.11}$$

$$\left[\Gamma_{jk}(\vec{n})\frac{\partial^2}{\partial s^2}+\rho\omega^2\delta_{jk}\right]\widehat{G}_{k4}=L_{j4}L_{44}^{-1}\delta(s) \tag{3.12}$$

$$\frac{\partial^2\widehat{G}_{4p}}{\partial s^2}=-L_{44}^{-1}L_{4k}\frac{\partial^2\widehat{G}_{kp}}{\partial s^2} \tag{3.13}$$

$$\frac{\partial^2\widehat{G}_{44}}{\partial s^2}=-L_{44}^{-1}\left[L_{4k}\frac{\partial^2\widehat{G}_{k4}}{\partial s^2}+\delta(s)\right] \tag{3.14}$$

in which

$$\Gamma_{jk}(\vec{n})=L_{jk}-L_{44}^{-1}L_{j4}L_{4k} \tag{3.15}$$

By transforming the coordinates to the bases of the eigenspaces of $\Gamma_{jk}(\vec{n})$, the system of inhomogeneous Eqs. (3.11)–(3.14) can be reduced to a system of uncoupled 1-D Helmholtz equations. The eigenvetors E_{jm} corresponding to the eigenvalues λ_m are defined by

$$\Gamma_{jk}(\vec{n})E_{km}=\lambda_m E_{jm}\quad(m=1,2,3) \tag{3.16}$$

It is noted that the summation convention does not applied the suffix, m, wherever λ_m, and later on, c_m and k_m appear. It is easily proved that $\Gamma_{jk}(\vec{n})$ is symmetric and positive matrix. Therefore, the eigenvetors are real value and can be taken as orthonormal bases, and the eigenvalues are positive real value. Hence, there exist

$$E_{jm}E_{jn}=E_{mj}E_{nj}=\delta_{mn} \tag{3.17}$$

The transformation of $\widehat{G}_{kp}$ to the new bases is given by

$$\widehat{G}^*_{mp}=E_{km}\widehat{G}_{kp} \tag{3.18}$$

The inverse transformation is then given by

$$\widehat{G}_{kp}=E_{kn}\widehat{G}^*_{np} \tag{3.19}$$

Both sides of Eq. (3.19) are multiplied by E_{jm}, then substituting (3.19) into (3.11) yield the result

$$\left[\lambda_m+\rho\omega^2\right]\widehat{G}^*_{mp}=-E_{pm}\delta(s)\quad(m=1,2,3) \tag{3.20}$$

The solution of above equations is given by Wang and Achenbach (1995)

$$\widehat{G}^*_{mp}=\frac{iE_{pm}}{2\rho c_m^2k_m}e^{ik_m|s|} \tag{3.21}$$

where c_m and k_m are the phase velocities and wave numbers defined by

$$c_m=\sqrt{\lambda_m/\rho},k_m=\omega/c_m \tag{3.22}$$

By applying Eq. (3.19) and (3.21), the solution of $\widehat{G}_{kp}$ is obtained as

$$\widehat{G}_{kp}=\sum_{m=1}^{3}\frac{iE_{km}E_{pm}}{2\rho c_m^2k_m}e^{ik_m|s|} \tag{3.23}$$

The inverse transform of (3.23) is

$$G_{kp}(\vec{x})=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{pm}}{2\rho c_m^2}\left[2\delta(\vec{n}\cdot\vec{x})+ik_me^{ik_m|\vec{n}\cdot\vec{x}|}\right]d\Omega(\vec{n}) \tag{3.24}$$

Obviously, the above formulations for Green's functions can be reduced to the results for the pure anisotropic elastic case (Willis, 1980). By use of Eqs. (3.16) and (3.17), it is easily verified that

$$\sum_{m=1}^{3}\frac{E_{km}E_{pm}}{\rho c_m^2}=\Gamma_{kp}^{-1}(\vec{n}) \tag{3.25}$$

Taking advantage of Eq. (3.13) and (3.23), the solution for $G_{4p}(\vec{x})$ is given by

$$G_{4p}(\vec{x})=\frac{-1}{8\pi^2}\int_{|\vec{n}|=1}\frac{\partial^2\widehat{G}_{4p}}{\partial s^2}\mathrm{d}\Omega(\vec{n})=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{pm}}{2L_{44}\rho c_m^2}\left[2\delta(\vec{n}\cdot\vec{x})+\mathrm{i}k\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n}) \tag{3.26}$$

Just as the process of seeking for the solutions for G_{kp} and $G_{4p}(\vec{x})$, similar mathematic operations can be applied for G_{k4} and G_{44}, the following results can be obtained:

$$G_{k4}(\vec{x})=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{jm}L_{j4}}{2L_{44}\rho c_m^2}\left[2\delta(\vec{n}\cdot\vec{x})+\mathrm{i}k_m\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n}) \tag{3.27}$$

and

$$G_{44}(\vec{x})=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{jm}L_{j4}}{2L_{44}\rho c_m^2}\left[2\delta(\vec{n}\cdot\vec{x})+\mathrm{i}k_m\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n}) \tag{3.28}$$

From Eqs. (3.24) and (3.26)–(3.28), it is noted that the integral representation for Green's displacements can be taken as the sum of the static contribution (when $\omega = 0$) and the correction by the dynamics. For the regular dynamic part, the integral can be evaluated numerically without any difficulty. For the singular part, the integral can be treated as the method proposed by Pan and Tonon (2000) for the static case. Corresponding representations for the operators appearing in Eqs. (2.25), (2.26), (2.29) and (2.30) are obtained as follows:

$$\left.\begin{aligned}
S_{kip}&=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{pm}n_i}{2\rho c_m^2}\left[2\delta'(\vec{n}\cdot\vec{x})-k_m^2\mathrm{sgn}(\vec{n}\cdot x)\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
S_{4ip}&=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{pm}n_i}{2L_{44}\rho c_m^2}\left[2\delta'(\vec{n}\cdot\vec{x})-k_m^2\mathrm{sgn}(\vec{n}\cdot x)\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
S_{ki4}&=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{lm}L_{l4}n_i}{2L_{44}\rho c_m^2}\left[2\delta'(\vec{n}\cdot\vec{x})-k_m^2\mathrm{sgn}(\vec{n}\cdot x)\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
S_{4i4}&=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{lm}L_{l4}n_i}{2L_{44}\rho c_m^2}\left[2\delta'(\vec{n}\cdot\vec{x})-k_m^2\mathrm{sgn}(\vec{n}\cdot x)\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})
\end{aligned}\right\} \tag{3.29}$$

and

$$\left.\begin{aligned}
(S_x)_{kjip}&=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{pm}n_in_j}{2\rho c_m^2}\left[2\delta''(\vec{n}\cdot\vec{x})-2k_m^2\delta(\vec{n}\cdot\vec{x})-\mathrm{i}k_m^3\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
(S_x)_{4jip}&=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{pm}n_in_j}{2L_{44}\rho c_m^2}\left[2\delta''(\vec{n}\cdot\vec{x})-2k_m^2\delta(\vec{n}\cdot\vec{x})-\mathrm{i}k_m^3\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
(S_x)_{kji4}&=\frac{1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{lm}L_{l4}n_in_j}{2L_{44}\rho c_m^2}\left[2\delta''(\vec{n}\cdot\vec{x})-2k_m^2\delta(\vec{n}\cdot\vec{x})-\mathrm{i}k_m^3\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})\\
(S_x)_{4ji4}&=\frac{-1}{8\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{4k}E_{km}E_{lm}L_{l4}n_in_j}{2L_{44}\rho c_m^2}\left[2\delta''(\vec{n}\cdot\vec{x})-2k_m^2\delta(\vec{n}\cdot\vec{x})-\mathrm{i}k_m^3\mathrm{e}^{\mathrm{i}k_m|\vec{n}\cdot\vec{x}|}\right]\mathrm{d}\Omega(\vec{n})
\end{aligned}\right\} \tag{3.30}$$

and

$$M_{Kp} = -\mathrm{i}\omega G_{Kp} \tag{3.31}$$

$$(M_x)_{Kip} = \mathrm{i}\omega S_{Kip} \tag{3.32}$$

and

$$(S_t)_{kiP} = -\mathrm{i}\omega S_{kiP} \tag{3.33}$$

$$(M_t)_{kp} = -\mathrm{i}\omega M_{kp} \tag{3.34}$$

Once Eqs. (2.27) and (2.28) have been solved for the scattering problem, the total field in (2.22) can be obtained. From equation (2.22), the total field can be considered as the sum of the incident field $\mathbf{u}^0$ and a scattered field $\mathbf{v}$. A scattered field $\mathbf{v}$ is

$$\mathbf{v} = -\mathbf{S}\boldsymbol{\tau} - \mathbf{M}\boldsymbol{\pi} \tag{3.35}$$

4. Scattering cross-section

In this section, we will focus on the scattering cross-section of the inclusion, both for its intrinsic interest and for its use in estimating attenuation in the inhomogeneous piezoelectric mediums. The scattering cross-section Q of inclusion is defined as the ratio of the total mean rate of outflow of energy associated with scattered field $\mathbf{v}$ to the mean energy flux in the direction $\mathbf{n}^0$ associated with the incident wave.

Assuming the extend stress σ_{iJ} is derived from the scattered field $\mathbf{v}$, the mean flux of energy associated with $\mathbf{v}$ has components

$$Y_i = -\frac{1}{4}\mathrm{i}\omega(\sigma_{iJ}\bar{v}_J - \bar{\sigma}_{iJ}v_J) \tag{4.1}$$

where the superposed bar denotes complex conjugation. The mean rate of energy radiation E out of a volume V is then defined by

$$E = \int_{\partial V} Y_i n_i^0 \mathrm{d}S \tag{4.2}$$

Using Gauss' theorem, we get

$$E = -\frac{1}{4}\mathrm{i}\omega\int_V (\sigma_{iJ,i}\bar{v}_J - \bar{\sigma}_{iJ,i}v_J + \sigma_{iJ}\bar{v}_{J,i} - \bar{\sigma}_{iJ}v_{J,i})\mathrm{d}\vec{x} \tag{4.3}$$

but

$$\sigma_{iJ,i} + \mathrm{i}\omega\pi_J + \omega^2\rho_{JK}v_K = 0 \tag{4.4}$$

and

$$\sigma_{iJ} = C_{iJKl}v_{K,l} + \tau_{iJ} \tag{4.5}$$

It follows, therefore, that

$$E = -\frac{1}{4}\mathrm{i}\omega\int_V (\tau_{iJ}\bar{v}_{J,i} - \bar{\tau}_{iJ}v_{J,i})\mathrm{d}\vec{x} - \frac{1}{4}\omega^2\int_V (\pi_j\bar{v}_j + \bar{\pi}_j v_j)\mathrm{d}\vec{x} \tag{4.6}$$

in above equation, the polarizations $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ are non-zero only over the volume V' occupied by the inclusion. Eq. (4.6) also shows that only the imaginary part of the produced $\tau_{iJ}\bar{v}_{J,i}$ and the real part contribute

$\pi_j \bar{v}_j$ to E. Taking into account Eq. (3.2), the mean energy flux in the direction $\mathbf{n}^0$ associated with the plane wave $\mathbf{u}^0$ can be expressed as follows

$$Y^0 = \frac{\omega k_0}{2} C_{iJKl} n_i^0 n_l^0 U_J U_K = \frac{\omega^3}{2k_0} \rho_{JK} U_J U_K = \frac{\rho\omega^3}{2k_0} U_j U_j \tag{4.7}$$

Then the scattering cross-section Q is given

$$Q = E/Y^0 \tag{4.8}$$

5. The Rayleigh limit

The system of equations (2.27) and (2.28) is difficult to solve explicitly but they can be simplified considerably in the low frequency range, or Rayleigh limit. Retention of lowest term reduces the equations to

$$[(\mathbf{C}^* - \mathbf{C})^{-1} + \mathbf{\Gamma}]_{iJQl} \tau_{iJ} = -\mathrm{i}kU_Q n_l^0 \quad \vec{x} \in V' \tag{5.1}$$

$$(\rho^* - \rho)^{-1} \pi_q = -\mathrm{i}\omega U_q \qquad \vec{x} \in V' \tag{5.2}$$

In above equations, if the inclusion's diameter is much smaller than the wavelengths of incident fields, $\boldsymbol{\tau}$ and $\boldsymbol{\pi}$ can be considered as constants over the inclusion. Γ^∞ in Eq. (5.1) is the static limit of operator $\mathbf{S}_x$, with kernel

$$\Gamma^\infty_{ijql} = \frac{1}{8\pi^2} \sum_{m=1}^{3} \int_{|\vec{n}|=1} \frac{E_{qm} E_{jm} n_i n_l}{\rho c_m} \delta''(\vec{n} \cdot \vec{x}) \mathrm{d}\Omega(\vec{n}) \tag{5.3}$$

$$\Gamma^\infty_{ij4l} = \frac{-1}{8\pi^2} \sum_{m=1}^{3} \int_{|\vec{n}|=1} \frac{L_{4q} E_{qm} E_{jm} n_i n_l}{L_{44} \rho c_m} \delta''(\vec{n} \cdot \vec{x}) \mathrm{d}\Omega(\vec{n}) \tag{5.4}$$

$$\Gamma^\infty_{i4ql} = \frac{-1}{8\pi^2} \sum_{m=1}^{3} \int_{|\vec{n}|=1} \frac{E_{qm} E_{jm} L_{j4} n_i n_l}{L_{44} \rho c_m} \delta''(\vec{n} \cdot \vec{x}) \mathrm{d}\Omega(\vec{n}) \tag{5.5}$$

$$\Gamma^\infty_{i44l} = \frac{1}{8\pi^2} \sum_{m=1}^{3} \int_{|\vec{n}|=1} \frac{L_{4q} E_{qm} E_{jm} L_{j4} n_i n_l}{L_{44} \rho c_m} \delta''(\vec{n} \cdot \vec{x}) \mathrm{d}\Omega(\vec{n}) \tag{5.6}$$

and reduces to a constant tensor $\mathbf{P}$; for an ellipsoid $\mathbf{x}^\mathrm{T}\mathbf{A}^\mathrm{T}\mathbf{A}\mathbf{x} = \lambda^2$, the expression of $\mathbf{P}$ is derived in Appendix B.

It follow from (5.1) that, to the lowest order, $\boldsymbol{\tau}$ is the static response to the incident field, while (5.2) shows that the momentum $\boldsymbol{\pi}$ is just that produced by inclusion being carried along by the incident wave. Hence, for an ellipsoidal inclusion, in the Rayleigh limit

$$\tau_{iJ} = -\mathrm{i}k \left[(\mathbf{C}^* - \mathbf{C})^{-1} + \mathbf{P} \right]^{-1}_{iJQl} U_Q n_l^0 \tag{5.7}$$

and for any inclusion

$$\pi_q = -\mathrm{i}\omega(\rho^* - \rho) U_q \tag{5.8}$$

Now we evaluate the scattering cross-section of an ellipsoidal inclusion, it can be obtained from (4.6) and (4.8), together with (3.35). Considering that $\boldsymbol{\tau}$ is constant over the inclusion, in the first integral term in (4.6) only the even part of $\bar{v}_{J,i}$ has a contribution. Therefore equation (3.35) shows that only imaginary and even parts of $\mathbf{S}_x$ and $\mathbf{M}_x$ contribute to Q. From (3.32) and (3.29), $\mathbf{M}_x$ is odd, while (3.30) shows that the relevant term in $\mathbf{S}_x$ is the one involving the exponential term. Furthermore, to the lowest order, the exponential term may be approximated to unit. This will simplify the results further. Similar considerations apply to the second integral term in (4.6), the asymptotic expression for Q is obtained

$$Q = \left(\frac{4\pi a_1 a_2 a_3}{3}\right)^2 \frac{\omega^4}{2Y^0}\left[\tau_{iJ}(\Delta\mathbf{S}_x)_{QliJ}\bar{\tau}_{lQ} + \pi_k \Delta M_{kp}\bar{\pi}_p\right] \tag{5.9}$$

for an ellipsoid with semi-axes a_1, a_2, a_3, where

$$\left.\begin{aligned}
(\Delta\mathbf{S}_x)_{qlij} &= \tfrac{1}{16\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{qm}E_{jm}n_i n_l}{\rho c_m^5}\,\mathrm{d}\Omega(\vec{n})\\
(\Delta\mathbf{S}_x)_{4lij} &= \tfrac{-1}{16\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{44}^{-1}L_{4k}E_{km}E_{jm}n_i n_l}{\rho c_m^5}\,\mathrm{d}\Omega(\vec{n})\\
(\Delta\mathbf{S}_x)_{qli4} &= \tfrac{-1}{16\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{44}^{-1}E_{qm}E_{km}L_{k4}n_i n_l}{\rho c_m^5}\,\mathrm{d}\Omega(\vec{n})\\
(\Delta\mathbf{S}_x)_{4li4} &= \tfrac{1}{16\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{L_{44}^{-1}L_{4k}E_{km}E_{pm}L_{p4}n_i n_l}{\rho c_m^5}\,\mathrm{d}\Omega(\vec{n})
\end{aligned}\right\} \tag{5.10}$$

and

$$\Delta M_{kp} = \frac{1}{16\pi^2}\sum_{m=1}^{3}\int_{|\vec{n}|=1}\frac{E_{km}E_{pm}}{\rho c_m^3}\,\mathrm{d}\Omega(\vec{n}) \tag{5.11}$$

6. Numerical examples

The scattering cross-section of a variety of inclusions in anisotropic elastic matrix has been considered by Willis (1980). In this section, the scattering cross-section is calculated for the composite, which is consisted of a single $BaTiO_3$-inclusion and a PZT-5H-matrix. The matrix and the inclusion are transversely isotropic piezoelectric material with the symmetry axis x_3, and their non-zero elements of material constants are $BaTiO_3$-inclusion:

$$c_{11}^* = 166\text{ GPa}, c_{33}^* = 162\text{ GPa}, c_{12}^* = 77\text{ GPa}, c_{13}^* = 78\text{ GPa}$$
$$c_{44}^* = 43\text{ GPa}, e_{31}^* = -4.4\text{ Cm}^{-2}, e_{33}^* = 18.6\text{ Cm}^{-2}, e_{15}^* = 11.6\text{ Cm}^{-2}$$
$$\varepsilon_{11}^* = 11.2\times 10^{-9}\text{ CN}^{-1}\text{m}^{-2}, \varepsilon_{33}^* = 12.6\times 10^{-9}\text{ CN}^{-1}\text{m}^{-2}, \rho^* = 5700\text{ Kgm}^{-3}$$

PZT-5H-matrix:

$$c_{11} = 126\text{ GPa}, c_{33} = 117\text{ GPa}, c_{12} = 55\text{ GPa}, c_{13} = 53\text{ GPa}$$
$$c_{44} = 35.5\text{ GPa}, e_{31} = -6.5\text{ Cm}^{-2}; e_{33} = 23.3\text{ Cm}^{-2}, e_{15} = 17.0\text{ Cm}^{-2}$$
$$\varepsilon_{11} = 15.1\times 10^{-9}\text{ CN}^{-1}\text{ m}^{-2}, \varepsilon_{33} = 13.0\times 10^{-9}\text{ CN}^{-1}\text{ m}^{-2}, \rho = 7500\text{ Kg m}^{-3}$$

In the examples, we suppose the incident wave propagates in the direction normal to the axis x_3 and the plane shear (longitudinal shear, SH) wave polarized in the x_3-direction is considered. Thus

$$\beta = \frac{\omega}{k_0} = \left(\frac{c_{44}\varepsilon_{11} + e_{15}^2}{\rho\varepsilon_{11}}\right)^{1/2} \tag{6.1}$$

$$\mathbf{n}^0 = [\cos\theta, \sin\theta, 0]^{\mathrm{T}} \tag{6.2}$$

where θ is the angle between the propagating direction of incident wave and the axis x_1.

Also, the energy flux relevant to the incident wave can be expressed as

$$Y^0 = \frac{\rho\omega^3}{2k_0} U_j U_j = \frac{\rho\beta\omega^2}{2} = \frac{\rho\beta^3 k_0^2}{2} \tag{6.3}$$

Here, U_j is taken as

$$[U_1, U_2, U_3] = [0, 0, 1] \tag{6.4}$$

U_4 is obtained by the following equation:

$$U_4 = \frac{e_{ikl} n_l n_i}{\varepsilon_{jp} n_j n_p} U_k = \frac{e_{15}}{\varepsilon_{11}} \tag{6.5}$$

We consider an inclusion having the shape of spheroid $(a, a, \varepsilon a)$, with the following special cases:

Case 1. A spherical inclusion, with $\varepsilon = 1$.
Case 2. A flat disc inclusion, with $\varepsilon = 1/50$.
Case 3. A flat disc cavity, with $\varepsilon = 1/50$, and $[(\mathbf{C}^* - \mathbf{C})^{-1} + \mathbf{P}]^{-1}$ reduces to $[\mathbf{P} - \mathbf{C}^{-1}]^{-1}$.
Case 4. A rigid disc, with $\varepsilon = 1/50$, and $[(\mathbf{C}^* - \mathbf{C})^{-1} + \mathbf{P}]^{-1}$ reduces to $\mathbf{P}^{-1}$.
Case 5. A long fibre, with $\varepsilon = 50$.
Case 6. A rigid fibre, with $\varepsilon = 50$, and $[(\mathbf{C}^* - \mathbf{C})^{-1} + \mathbf{P}]^{-1}$ reduces to $\mathbf{P}^{-1}$.

The numerical results of the scattering cross-section for the special cases are represented in Table 1, which are normalized with respect to $k_0^4 a^6$. It is worth to point out that the results are independent of the angle θ because the matrix is transversely isotropic.

In order to show the validity and feasibility of the relevant formulations, such as Eqs. (2.22), (5.7)–(5.9), we let the piezoelectric materials' piezoelectric moduli and the permittivity equal to zero. In the piezoelectric case, the lowercase and uppercase subscripts in Eqs. (2.22), (5.7)–(5.9) take on the range 1–3 and 1–4, respectively. When both the lowercase and uppercase subscripts in these equations only take on the range 1–3, it means that the piezoelectric moduli and the permittivity tend to zero, i.e, These formulations can easily be reduced to the corresponding formulations in Willis' paper (1980). The numerical results for corresponding purely elastic cases are presented in Table 2. The numerical results in Tables 1 and 2 show that the cross-section Q depends upon the properties of the inclusions and matrix. The piezoelectric materials' piezoelectric moduli and the permittivity may increase the Q that is higher than this in pure elastic case. The

Table 1
The piezoelectric matrix

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
3.083464×10^{18}	7.702814×10^{12}	9.532695×10^{12}	9.156199×10^{12}	9.294928×10^{21}	5.028276×10^{23}

Table 2
The elastic matrix

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
8.738551×10^{17}	3.611364×10^{12}	5.028642×10^{12}	4.758279×10^{12}	7.119862×10^{20}	9.524664×10^{22}

cross-section Q is one of factors governing the wave attenuation. It is noted that the Q showed in Table 1 is the cross-section of a single inclusion and the energy flux term is calculated from the elastic moduli, piezoelectric moduli and permitivity of matrix. When the formulation of the Q is used in composite piezoelectric material with a lot of inclusions, those material parameters should be the overall moduli.

7. Conclusions

The goal of this paper is to provide the dynamic general solution for an infinite, piezoelectric medium containing a single piezoelectric, ellipsoidal inclusion. The scattering problem is formulated in terms of integral equations (2.22), whose kernels are obtained from the Green's functions for a comparison body. The novel feature of Eq. (2.22) is the introduction of stress polarization and momentum polarization. In this paper, the Random transform method is used for the dynamic Green's functions. The most interesting advantage of this transform is that it reduces a three-dimensional partial differential equation to a one-dimensional partial differential equation. After the 1-D time harmonic wave problem is solved, the 3-D Green's function follows from the application of inverse transform. Finally, the asymptotic solution for the scattering cross-section is derived in the Rayleigh limit.

Appendix A

Consider function $f(\vec{x})$ defined in R^3, the Radon transform of $f(\vec{x})$ is defined as

$$\hat{f}(s,\vec{n}) = R[f(\vec{x})] = \int f(\vec{x})\delta(s-\vec{n}\cdot\vec{x})\mathrm{d}\vec{x} \tag{A.1}$$

where $\vec{n}$ is a unite vector and $\delta(\,)$ is the one-dimensional Dirac delta. The Radon transform is an integration of $f(\vec{x})$ over all planes defined by $\vec{n}\cdot\vec{x}=s$.

The inverse Radon transform defined as

$$f(\vec{x}) = R^*[\hat{f}''] = -\frac{1}{8\pi^2}\int_{|\vec{n}|=1}\hat{f}''(\vec{n}\cdot\vec{x},\vec{n})\mathrm{d}\Omega(\vec{n}) \tag{A.2}$$

in which

$$\hat{f}''(\vec{n}\cdot\vec{x},\vec{x}) = \left.\frac{\partial^2\hat{f}(s,\vec{n})}{\partial s^2}\right|_{s=\vec{n}\cdot\vec{x}} \tag{A.3}$$

Appendix B

For the static problem in the piezoelectric medium, the infinite-body Green's function **G** satisfies the following equation:

$$C_{iJKl}G_{KP,li}(\vec{x}) + \delta_{JP}\delta(\vec{x}) = 0 \tag{B.1}$$

A convenient representation for **G** is easiest obtained by employing the plane-wave expansion

$$\delta(\vec{x}) = \int_{|\vec{n}|=1}\delta''(\vec{n}\cdot\vec{x})\mathrm{d}S \tag{B.2}$$

We observe that

$$G^{*}_{KP}(\vec{n}\cdot\vec{x}) = -(C_{iJKl}n_l n_i)^{-1}\delta_{JP}\delta(\vec{n}\cdot\vec{x}) \tag{B.3}$$

satisfies the equation

$$C_{iJKl}G^{*}_{KP,li}(\vec{n}\cdot\vec{x}) + \delta_{JP}\delta''(\vec{n}\cdot\vec{x}) = 0 \tag{B.4}$$

Hence, the expression for G is given

$$G_{JK}(\vec{x}) = \frac{1}{8\pi^2}\int_{|\vec{n}|=1}(C_{iJKl}n_i n_l)^{-1}\delta(\vec{n}\cdot\vec{x})\mathrm{d}S \tag{B.5}$$

and

$$\Gamma^{\infty}_{pJKq}(\vec{x}) = -G_{JK,pq}(\vec{x}) = -\frac{1}{8\pi^2}\int_{|\vec{n}|=1} n_q n_p L^{-1}_{JK}(\vec{n})\delta''(\vec{n}\cdot\vec{x})\mathrm{d}S \tag{B.6}$$

Now we introduce the constant tensor $\mathbf{P}$ defined by Eshelby (1957) into piezoelectric ellipsoidal inclusion ($\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} < \lambda^2$) problem, the constant tensor $\mathbf{P}$ is defined by

$$\mathbf{P} = \int_{\lambda<a}\Gamma^{\infty}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{B.7}$$

is independent of the value of $a > 0$. so that

$$P_{pJKq} = -\frac{1}{8\pi^2}\int_{|\vec{n}|=1} n_p n_q L^{-1}_{JK}(\vec{n})\mathrm{d}S\int_{\lambda<1}\delta''(\vec{n}\cdot\vec{x})\mathrm{d}\vec{x} = \frac{1}{4\pi}|\mathbf{A}|^{-1}\int_{|\vec{n}|=1}\frac{n_p n_q L^{-1}_{JK}(\vec{n})\mathrm{d}S}{\left[\mathbf{n}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{n}\right]^{3/2}} \tag{B.8}$$

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